Ball Up/Down

If you throw a small ball vertically upward in real air with drag, does it take longer to go up or come down?

Solution by Ilkka Mäkinen

Let us start by considering the forces on the ball. There are two forces: the weight of the ball and the air resistance. If we assume that the air resistance is proportional to the ball's velocity, the equation of motion for the ball is

$$ma = -mg - kv$$

or

$$my'' = -mg - ky'$$

where ' indicates differentiation with respect to time. Dividing this by m and letting $k/m = A$, we have

$$y'' = -Ay' - g$$

which is a second-order differential equation of y. We first solve it for the velocity, y', by putting $y'' = dy'/dt$, and separating the variables:

$$dy'/(Ay' + g) = -dt$$

Integrating, we get

$$(1/A) \ln(Ay' + g) = -t + C_1$$

whence

$$y' = C_2 e^{-At} - (g/A)t$$

and

$$y(t) = C_3 + C_4 e^{-At} - (g/A)t$$

Now there are two unknown constants in the solution, so we need two initial conditions. We can choose $y(0)=0$ and $y'(0)=v_0$, i.e. the ball's initial height is zero and its initial velocity is v_0. From these conditions, we have

$$C_3 + C_4 e^0 + 0 = 0 \qquad \text{and} \qquad -C_4 A e^0 - (g/A) = v_0$$

Therefore

$$C_3 = (v_0/A) + (g/A^2) \qquad \text{and} \qquad C_4 = -(v_0/A) - (g/A^2)$$

Putting these back into the equation for y(t), we get an expression for the ball's height as a function of time:

$$y(t) = [(v_0/A) + (g/A^2)] (1 - e^{-At}) - (g/A) t$$

Now we proceed by finding the time in which the ball reaches the highest point, and proving that when twice this time has elapsed, the ball hasn't yet reached the ground (i.e. if the ball reaches the highest point in T seconds, we want to prove that y(2T)>0 for all (positive) values of v_0 and A).

The first part is easy. In the highest point, the velocity of the ball is zero, so we take the derivative of the function y(T), set it equal to zero and solve it for T:

$$y'(T) = A[(v_0/A) + (g/A^2)] e^{-AT} - (g/A) = 0$$

whence

$$e^{-AT} = g / \{A^2[(v_0/A) + (g/A^2)]\} = g/(v_0A + g)$$

and

$$T = -1/A \ln [g/(v_0A + g)]$$

Now, the ball's height after twice this time is

$$y(2T) = [(v_0/A) + (g/A^2)] (1 - e^{-A\{-2/A \ln [g/(v0A + g)]\}}) - (g/A) \{-2/A \ln [g/(v_0A + g)]\}$$

$$= [(v_0/A) + (g/A^2)] [1 - g^2/(v_0A + g)^2] + (2g/A^2) \ln [g/(v_0A + g)]$$

When we calculate the product on the left, multiply all the terms so that they have the same denominator, and reduce the resulting expression to the simplest possible form, we get

$$y(2T) = (v_0/A) [1 + g/(v_0A + g)] + (2g/A^2) \ln [g/(v_0A + g)]$$

Therefore, to finish the solution, we need to prove that the inequality

$$(v_0/A) [1 + g/(v_0A + g)] + (2g/A^2) \ln [g/(v_0A + g)] > 0$$

or

$$(v_0A/2g) [1 + g/(v_0A + g)] + \ln [g/(v_0A + g)] > 0$$

is true for all positive values of A and v_0. To do this, we put $x = 1 + v_0A/g$, when the inequality is the same as

$$\tfrac{1}{2} (x - 1/x) > \ln x$$

for x > 1. When x = 1 both sides of the inequality equal 0. Therefore the inequality must hold true if the left side is always increasing faster than the right side – that is, if

$$(d/dx) \tfrac{1}{2} (x - 1/x) > (d/dx) \ln x$$

for x > 1, or, taking the derivatives and multiplying both side by 2*x*,

$$x + 1/x > 2$$

which is certainly true for x > 1.