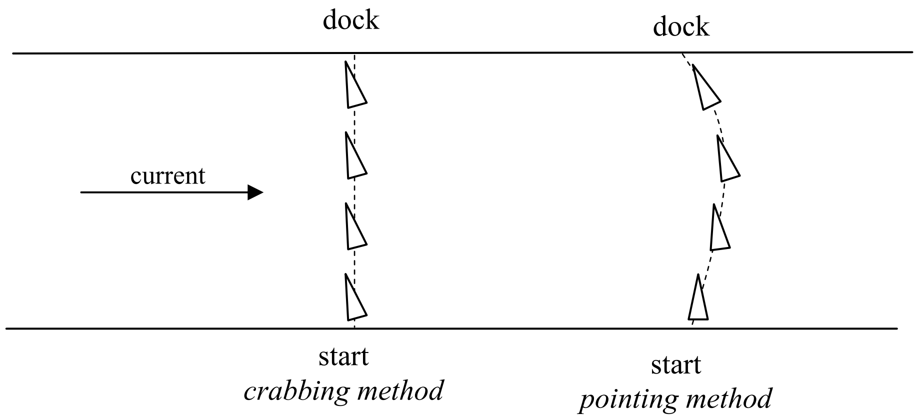
boat time

Suppose you are anchored near the shore of a channel in which there is steady current, and you are going to run your (motor)boat at constant throttle to a dock directly across the channel on the opposite shore. There are two ways one might steer the boat to the dock:

- *the crabbing method*: steer a steady course with the nose of the boat pointed somewhat upstream, so the boat maintains a fixed orientation and crabs in a straight line across the channel
- *the pointing method*: keep the nose of the boat pointed directly at the dock



Which method gets the boat to the dock faster, and by how much? (Assume the boat runs at a constant speed relative to the water, which is faster than the speed of the current relative to the shore.)

Michael A. Gottlieb's Solution

Notation:

L is the width of the channel.

$\mathbf{v}_{sw}$ is the velocity of the *shore* relative to the *water.*
$\mathbf{v}_{ws}$ is the velocity of the *water* relative to the *shore.*

So, $\mathbf{v}_{ws} = -\mathbf{v}_{sw}$.

Let $|\mathbf{v}_{ws}| = s_w = -|\mathbf{v}_{sw}|$ *with* $s_w > 0$.

(x_w, y_w) is the boat's coordinates in the frame of the *water*.
(x_s, y_s) is the boat's coordinates in the frame of the *shore*.

The origin of the shore's frame is the boat's destination; the origin of both frames coincide at time $t = 0$. The positive y-axis points in the direction of the current, and the positive x axis points in the direction of the boat's starting place.

The transformation between frames is:

$y_w = y_s - s_w\, t$
$x_w = x_s$

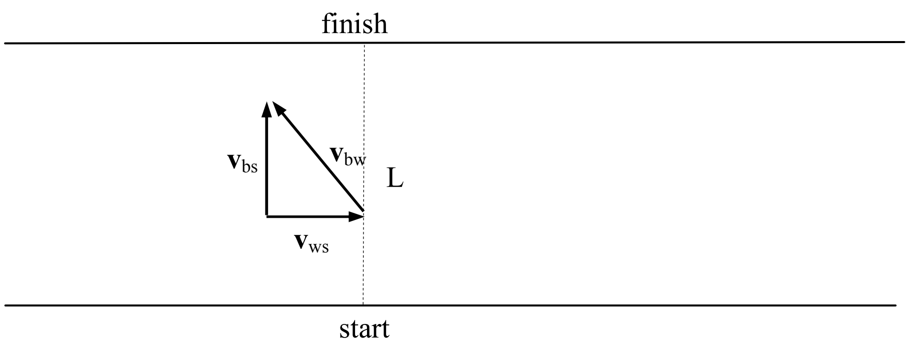
$\mathbf{v}_{\text{bw}} = (\frac{dx_w}{dt}, \frac{dy_w}{dt})$ is the velocity of the *boat* relative to the *water*.
$\mathbf{v}_{\text{bs}} = (\frac{dx_s}{dt}, \frac{dy_s}{dt})$ is the velocity of the *boat* relative to the *shore*.

It is given that the magnitude of $\mathbf{v}_{\text{bw}}$ is constant, so

Let $|\mathbf{v}_{\text{bw}}| = s_b$ *with* $s_b > 0$.

The Crabbing Method

For the crabbing method we have the following situation:

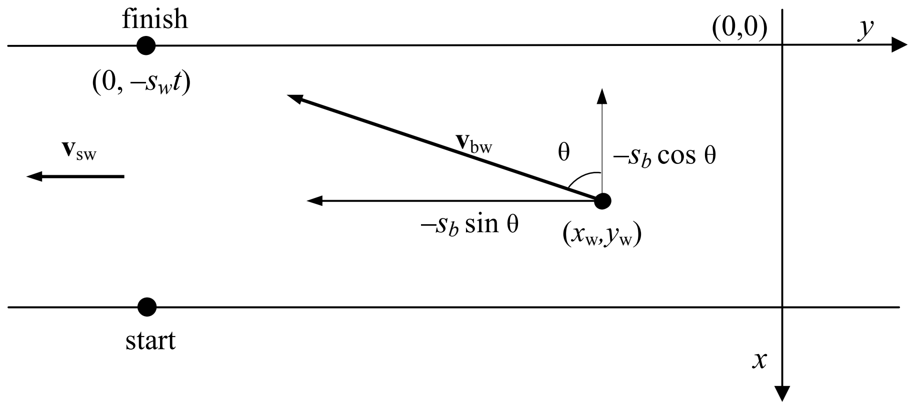


So, $\left|\mathbf{v}_{\text{bs}}\right| = \sqrt{\mathbf{v}_{\text{bw}}^2 - \mathbf{v}_{\text{ws}}^2} = \sqrt{s_{\text{b}}^2 - s_{\text{w}}^2}$, and therefore the time it takes the boat to cross the channel using the crabbing method is

$$\frac{\text{L}}{\sqrt{s_{\text{b}}^2 - s_{\text{w}}^2}} = \frac{\text{L}/\text{s}_\text{b}}{\sqrt{1-\left(\dfrac{\text{s}_\text{w}}{\text{s}_\text{b}}\right)^2}}.$$

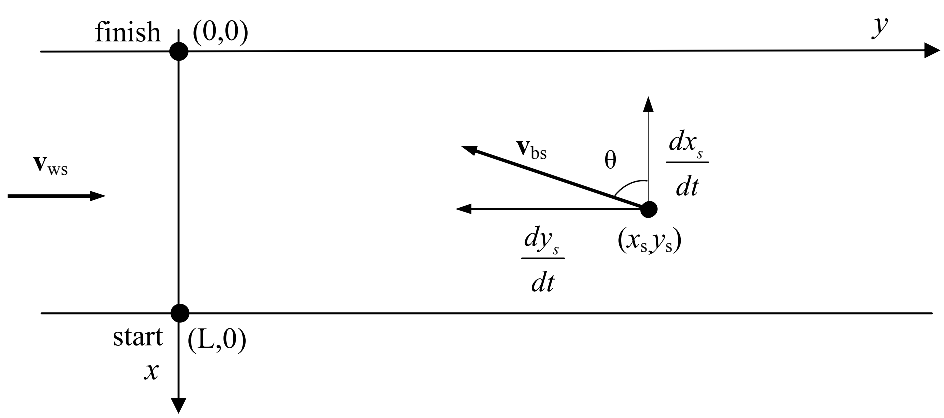
The Pointing Method

Consider the situation viewed from the frame of the water at time t:



We have tan θ = $(y_w + s_w t) / x_w$, and taking components we find the boat's speed parallel and perpendicular to the shore (relative to the water): $-s_b \sin\theta$ and $-s_b \cos\theta$, respectively.

Now consider the situation viewed from the frame of the shore at time t:



First note that θ, which depends only on the boat's position relative to the destination, is the same in both illustrations. In terms of the frame of the shore, tan θ = y_s / x_s. Then observe that the speed of the boat in the y direction has to be s_w more in the frame of the shore than it is in the frame of the water, while the speed of the boat in the x direction is the same in both frames.

So we have,

$$\frac{dx_s}{dt} = -s_b \cos\theta$$
$$\frac{dy_s}{dt} = -s_b \sin\theta + s_w \tag{1}$$

And now we are ready to solve for the path of the boat. From here forward I put everything in the frame of the shore, and I drop the subscripts:

$$x' = -s_b \cos\theta \qquad y' = -s_b \sin\theta + s_w \tag{2}$$

[Before I procede, here's a simple sanity check. We have…

$$(x')^2 = s_b^2 \cos^2\theta \qquad (y' - s_w)^2 = s_b^2 \sin^2\theta$$

$$\therefore\ (x')^2 + (y' - s_w)^2 = s_b^2$$

… which is right, by the Pythagorean theorem, because $(x', y' - s_w)$ are the boat's (x, y) velocity components relative to the water, and s_b is the boat's speed relative to the water.]

Since $\dfrac{dy}{dx} = \dfrac{y'}{x'}$, we have, from (2):

$$\frac{y'}{x'} = \frac{-s_b \sin\theta + s_w}{-s_b \cos\theta} = \tan\theta - \frac{s_w}{s_b \cos\theta}$$

$$= \left(\frac{y}{x}\right) - \left(\frac{s_w}{s_b}\right)\frac{1}{\cos\theta}$$

But, since $\cos\theta = \dfrac{x}{\sqrt{x^2 + y^2}}$, the curve the boat follows satisfies:

$$\frac{dy}{dx} = \left(\frac{y}{x}\right) - \left(\frac{s_w}{s_b}\right)\frac{\sqrt{x^2 + y^2}}{x}$$

with boundary conditions y[L] = 0 and y[0]=0. Rearranged this is:

$$\frac{s_b\left(y - x\dfrac{dy}{dx}\right)}{\sqrt{x^2 + y^2}} = s_w$$

I used Mathematica to solve this differential equation:

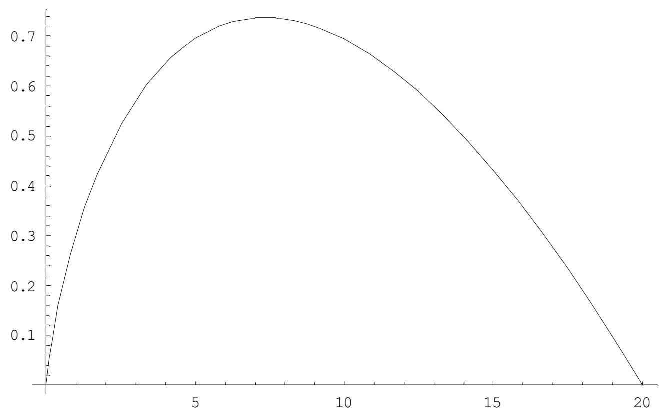
```
DSolve[{Sb*(y[x] - x*y'[x])/Sqrt[x^2 + y[x]^2] == Sw, y[0] == 0, y[L] == 0}, y[x], x]
```

$$\left\{y[x] \to \frac{1}{2}\left(L^{Sw} x^{Sb-Sw}\right)^{-1/Sb}\left(-x^2 + \left(L^{Sw} x^{Sb-Sw}\right)^{2/Sb}\right),\ y[x] \to \frac{1}{2}\left((-L)^{Sb} L^{-Sb+Sw} x^{Sb-Sw}\right)^{-1/Sb}\left(-x^2 + \left((-L)^{Sb} L^{-Sb+Sw} x^{Sb-Sw}\right)^{2/Sb}\right)\right\}$$

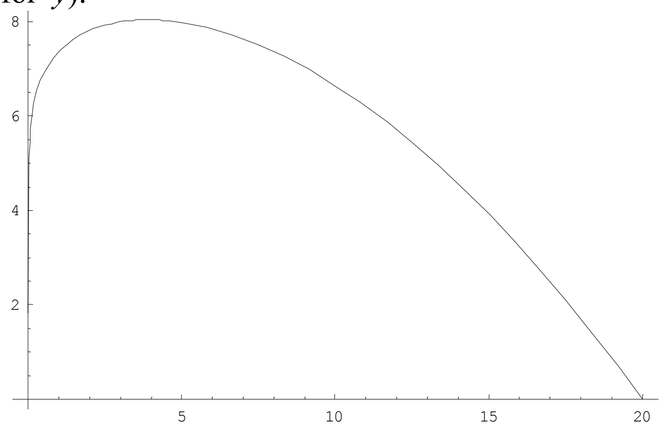
Only the first of the two solutions is positive real. Simplifying yields

$$y[x] = \frac{1}{2} L^{\frac{Sw}{Sb}} x^{1-\frac{Sw}{Sb}} - \frac{1}{2} L^{-\frac{Sw}{Sb}} x^{1+\frac{Sw}{Sb}}$$

Here's a graph of the boat's path for L=20, s_b=20, s_w=2 – a weak current (Note: different scales for *x* and *y*):



And here's the boat's path for L=20, s_b=20, s_w=18 – a strong current (at yet another scale for *y*):



It has been shown that (*in the frame of the shore*):

$$\frac{dx}{dt} = -s_b \cos\theta \quad \text{with} \quad \cos\theta = \frac{x}{\sqrt{x^2+y^2}},$$

and therefore:

$$\frac{dt}{dx} = -\frac{1}{s_b \cos\theta} = -\frac{\sqrt{x^2+y^2}}{s_b x}.$$

It has also been shown that

$$y(x) = \tfrac{1}{2}L^{\frac{s_w}{s_b}} x^{\left(1-\frac{s_w}{s_b}\right)} - \tfrac{1}{2}L^{-\frac{s_w}{s_b}} x^{\left(1+\frac{s_w}{s_b}\right)},$$

and therefore:

$$\frac{dt}{dx} = -\frac{\sqrt{x^2 + \left(\tfrac{1}{2}L^{\frac{s_w}{s_b}} x^{\left(1-\frac{s_w}{s_b}\right)} - \tfrac{1}{2}L^{-\frac{s_w}{s_b}} x^{\left(1+\frac{s_w}{s_b}\right)}\right)^2}}{s_b x} = -\frac{L^{\frac{s_w}{s_b}} x^{-\frac{s_w}{s_b}} + L^{-\frac{s_w}{s_b}} x^{\frac{s_w}{s_b}}}{2s_b}$$

Integrating with respect to *x* and settig *t*(L)=0 yields

$$t(x) = \frac{L\,s_b}{s_b^2 - s_w^2} - \frac{L^{\frac{s_w}{s_b}} x^{\left(1-\frac{s_w}{s_b}\right)}}{2(s_b - s_w)} - \frac{L^{-\frac{s_w}{s_b}} x^{\left(1+\frac{s_w}{s_b}\right)}}{2(s_b + s_w)},$$

and thus *t*(0), the time it takes the boat to cross the channel using the pointing method is

$$\frac{L\,s_b}{s_b^2 - s_w^2} = \frac{L/s_b}{1-\left(\frac{s_w}{s_b}\right)^2}.$$

Comparison of crossing times

crabbing method: $$\frac{L/s_b}{\sqrt{1-\left(\frac{s_w}{s_b}\right)^2}}$$

pointing method: $$\frac{L/s_b}{1-\left(\frac{s_w}{s_b}\right)^2}$$

Since $1-\left(\frac{s_w}{s_b}\right)^2 < 1$, the crabbing method is faster.