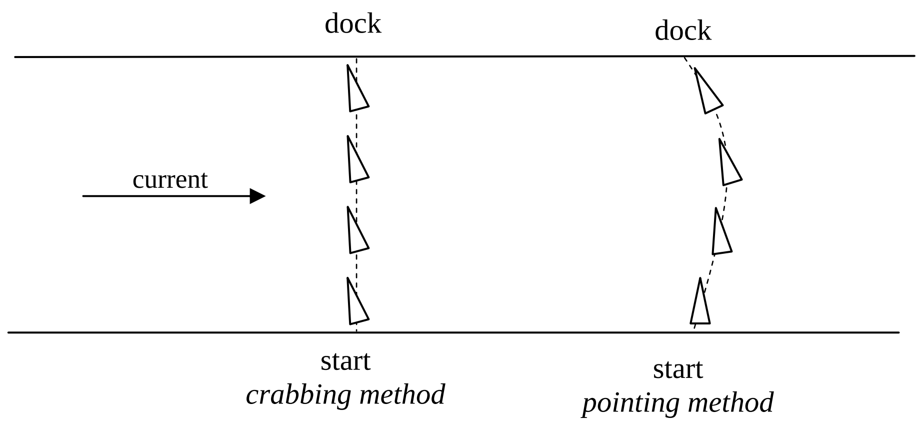
boat time

Suppose you are anchored near the shore of a channel in which there is steady current, and you are going to run your (motor)boat at constant throttle to a dock directly across the channel on the opposite shore. There are two ways one might steer the boat to the dock:

- *the crabbing method*: steer a steady course with the nose of the boat pointed somewhat upstream, so the boat maintains a fixed orientation and crabs in a straight line across the channel
- *the pointing method*: keep the nose of the boat pointed directly at the dock



Which method gets the boat to the dock faster, and by how much? (Assume the boat runs at a constant speed relative to the water, which is faster than the speed of the current relative to the shore.)

<u>Solution by Riccardo Borghi</u>

We denote by $\boldsymbol{v}_c$ the current speed and by $\boldsymbol{v}_b$ the boat speed with respect to the water. The speed, say $\boldsymbol{v}$, of the boat with respect to the dock will then be given by

$$\boldsymbol{v} = \boldsymbol{v}_c + \boldsymbol{v}_b \qquad (1)$$

Moreover, we introduce the ratio, say η, between the modula of $\boldsymbol{v}_c$ and $\boldsymbol{v}_b$ as

$$\eta = \frac{v_b}{v_c} \qquad (2)$$

which will be supposed to be greater than one. Finally, we denote by D the distance between the starting point, say P, and the arrival point, say O. We will analyze separately the two methods proposed to get the dock.

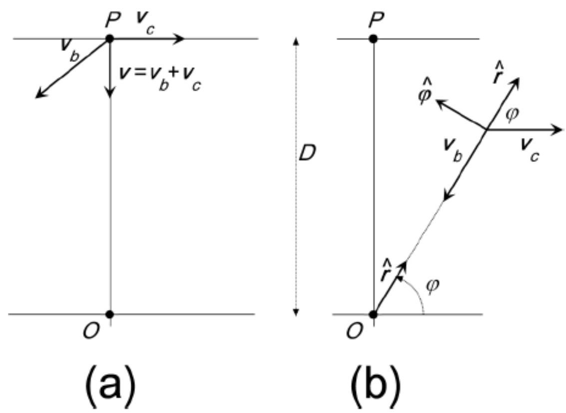


Fig. 1

- *Crabbing method*. The situation is depicted in Fig. 1a. In this case the speed $\boldsymbol{v}$ is a constant vector directed along the line connecting the points *P* and *O*. Its modulus v is then simply given by Pitagora's theorem and the navigation time, say T_1, turns out to be

$$T_1 = \frac{D}{v_c} \frac{1}{\sqrt{\eta^2 - 1}} \tag{3}$$

- *Pointing method.* To find the navigation time, say T_2, we will first determine the boat trajectory for a given η. To find it, we use a *polar* reference frame, say (r,φ), having the origin at the arrival point *O*. The situation is depicted in Fig. 1b. With respect to such a reference frame, the boat speed $\boldsymbol{v}$ will be expressed as

$$\boldsymbol{v} = \dot{r}\,\hat{\boldsymbol{r}} + r\,\dot{\varphi}\,\hat{\boldsymbol{\varphi}} \tag{4}$$

  As far as the r.h.s. of Eq. (1) is concerned, always from Fig. 1b we have

$$\boldsymbol{v}_b = -v_b\,\hat{\boldsymbol{r}} \tag{5a}$$

$$\boldsymbol{v}_c = v_c\,\cos\varphi\,\hat{\boldsymbol{r}} - v_c\,\sin\varphi\,\hat{\boldsymbol{\varphi}} \tag{5b}$$

  so that we have to solve the following system:

$$\dot{r} = v_c\,\cos\varphi - v_b \tag{6a}$$

$$r\dot{\varphi} = -v_c\,\sin\varphi \tag{6b}$$

with the initial conditions $r(0)=D$ and $\varphi(0)=\pi/2$. The boat trajectory can be obtained by employing a trick used by A. Sommerfeld to determine the form of Keplerian orbits.[1] In particular, on dividing side by side Eqs. (6a) and (6b), we have

$$\frac{1}{r}\,\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{\eta - \cos\varphi}{\sin\varphi} \tag{7}$$

that can be solved at once simply by separating variables r and φ.

[1] A. Sommerfeld, *Lectures on Theoretical Physics. I. Mechanics* (Academic Press, 1970)

Note that the boat trajectory corresponds to the boundary condition $r(\pi/2)=D$, which gives

$$\int_D^r \frac{\mathrm{d}r}{r} = \int_{\pi/2}^{\varphi} \frac{\eta - \cos\varphi}{\sin\varphi}\,\mathrm{d}\varphi \qquad (8)$$

Both integrals are expressed via elementary functions and give[2]

$$r(\varphi) = D\,\frac{\left(\tan\frac{\varphi}{2}\right)^{\eta}}{\sin\varphi} \qquad (9)$$

To find T_2 it is now sufficient to substitute from Eq. (9) into Eq. (6b), to separate variables φ and t and to integrate side by side. We have, on taking into account that $\varphi(0)=\pi/2$ and $\varphi(T_2)=0$,

$$T_2 = \frac{D}{v_c}\int_0^{\pi/2} \frac{\left(\tan\frac{\varphi}{2}\right)^{\eta}}{\sin^2\varphi}\,\mathrm{d}\varphi \qquad (10)$$

On evaluating the last integral[3] we have

$$T_2 = \frac{D}{v_c}\,\frac{\eta}{\eta^2-1} \qquad (11)$$

which, together with Eq. (3), gives at once

$$\frac{T_2}{T_1} = \frac{\eta}{\sqrt{\eta^2-1}} = \frac{1}{\sqrt{1-\frac{v_c^2}{v_b^2}}} \qquad (14)$$

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[2] The following indefinite integral should be taken into account:

$$\int \frac{\eta-\cos\varphi}{\sin\varphi}\,\mathrm{d}\varphi = \eta\log\tan\frac{\varphi}{2} - \log\sin\varphi$$

[3] We have

$$\int_0^{\pi/2} \frac{\left(\tan\frac{\varphi}{2}\right)^{\eta}}{\sin^2\varphi}\,\mathrm{d}\varphi = \frac{1}{2}\int_0^{\pi/2} \frac{\left(\tan\frac{\varphi}{2}\right)^{\eta}}{\sin^2\frac{\varphi}{2}}\,\frac{\mathrm{d}\varphi}{2\cos^2\frac{\varphi}{2}}$$

that, on making the substitution $u=\tan\varphi/2$ and on expressing sine and cosine via tangent, takes on the form

$$\frac{1}{2}\int_0^1 u^{\eta-2}(1+u^2)\,\mathrm{d}u = \frac{1}{2}\int_0^1 (u^{\eta-2}+u^{\eta})\,\mathrm{d}u = \frac{1}{2}\left(\frac{1}{\eta-1}+\frac{1}{\eta+1}\right) = \frac{\eta}{\eta^2-1}$$