bursting shell

This problem was contributed to the Feynman Lectures Website by Sukumar Chandra.

A shell flying with velocity 500 m/s bursts into three identical fragments so that the kinetic energy of the system increases 1.5 times. What maximum velocity can one of the fragments obtain?

Solution by Riccardo Borghi

Let $\boldsymbol{V}$ be the initial velocity of the shell with respect to a ‘‘fixed’’ reference frame. We want to propose a solution of the problem by using a reference frame moving with the shell velocity $\boldsymbol{V}$, in which the center of mass (CM henceforth) of the three fragments is at rest. In such a reference frame, on denoting $\boldsymbol{U}_i$ (i=1,2,3) the velocities of the fragments, the momentum conservation law gives

$$\boldsymbol{U}_1 + \boldsymbol{U}_2 + \boldsymbol{U}_3 = 0 \tag{1}$$

while the total kinetic energy of the fragments must equal the sole energy increment. In particular, on denoting ε the fraction of the initial energy corresponding to such increment,[1] the energy conservation law gives

$$U_1^2 + U_2^2 + U_3^2 = 3\,\varepsilon\, V^2 \tag{2}$$

Equation (1) has an immediate geometrical interpretation: the three velocities $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, and $\boldsymbol{U}_3$ must form a triangle, as shown in Fig. 1, where α denotes the angle between the directions of $\boldsymbol{U}_1$ and $\boldsymbol{V}$.

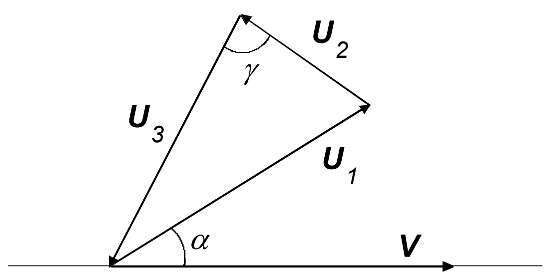

Figure 1

In order to maximize the speed, for example of the fragment ‘1’, in the fixed reference frame, we have first to make $\boldsymbol{U}_1$ and $\boldsymbol{V}$ parallel (by rotating the triangle in Fig. 1 to set $\alpha = 0$) and then to maximize the modulus U_1: the speed of the fragment will be then $V + U_1$. To this end we recast Eq. (2) as

$$U_1^2 = 3\,\varepsilon\, V^2 - (U_2^2 + U_3^2) \tag{3}$$

from which it follows that, in order to maximize U_1^2, the quantity $U_2^2+U_3^2$ has to be made as small as possible. However, such a minimization problem is *constrained*. To show this, we apply the law of cosines to Eq. (1), which gives

$$U_1^2 = U_2^2 + U_3^2 - 2\,U_2\,U_3\,\cos\gamma \tag{4}$$

that, together with Eq. (3), provides the following constraint for the *three* free parameters U_2, U_3, and γ:

$$U_2^2 + U_3^2 - U_2\,U_3\,\cos\gamma = \frac{3}{2}\,\varepsilon\, V^2 \tag{5}$$

To solve the constrained minimization problem we first let $U_2 = u + v$ and $U_3 = u - v$, where u and v are two auxiliary variables in terms of which Eq. (5) transforms into

$$(2 - \cos\gamma)\,u^2 + (2 + \cos\gamma)\,v^2 = \frac{3}{2}\,\varepsilon\, V^2 \tag{6}$$

and the quantity to be minimized into $u^2 + v^2$, disregarding an unessential factor 2. On interpreting (u, v) as the Cartesian coordinate of a point in a two-dimensional space, we see that our minimization problem has a

[1] For our problem ε=1/2.

simple geometrical interpretation: *find the point on the ellipse described by Eq. (6) whose distance from the center of the ellipse is minimum*.

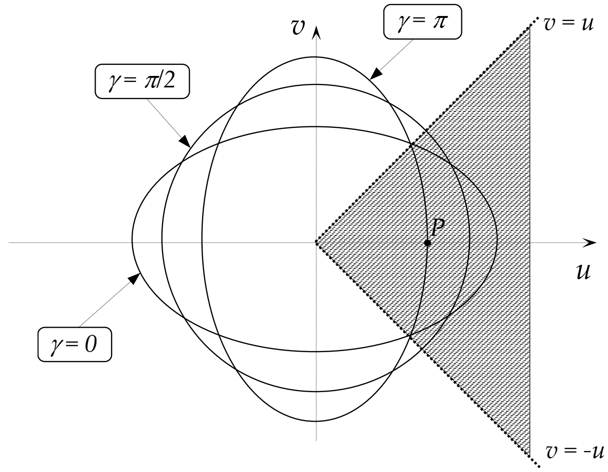

Figure 2

In Fig. 2 a graphical solution of the above problem is presented: first of all we note from Eq. (6) that the shape of the ellipse depends on γ. In particular, the ellipses corresponding to $\gamma = 0$, $\gamma = \pi/2$, and $\gamma = \pi$ have been plotted for clarity. In the same figure the grey zone corresponds to the pairs (u,v) satisfying the condition $-u < v < u$: such pairs are the sole physically acceptable.[2] Then it is immediate to convince that the point P corresponding to $\gamma = \pi$, $u^2=\varepsilon V^2/2$, and v=0 represents the solution of the problem. Accordingly, we have $U_2^2+U_3^2 = 2u^2 = \varepsilon V^2$ and, from Eq. (3), $U_1^2=2\varepsilon V^2$; in this way the *maximum* speed of the fragment '1' in the ''fixed'' reference turns out to be

$$V\left(1+\sqrt{2\varepsilon}\right) \qquad (7)$$

For the present case V=500 m/s and ε=1/2, so that Eq. (7) gives a maximum speed of 1000 m/s.

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[2] Remember that U_2 and U_3 must be both *positive*.