five pills

Suppose you are taking one each of 5 different types of pills every day but you don't like having to open and close 5 different bottles, so at the beginning of each (30-day) month you put 30 of each type of pill into one big bottle. When it is time to take your pills, you draw them out of the big bottle one at a time until you have (at least) one of each type. On the last day of the month you will draw exactly 5 pills and they will all be different (because that's all that's left in the bottle), but on other days you will generally have to draw more than 5 pills in order to have (at least) one of each type. So, the question is: On the first day of each month (when there are 150 pills in the bottle), how many pills, on average, must you draw from the bottle in order to have (at least) one of each?

Michael Gottlieb's Solution

Let there be m types of pills, n of each type, so that there are mn pills in the bottle. I assume that all pills in the bottle are equally likely to be drawn.

The number of combinations of k pills which lacks pills of at least one type is:

$$L(m,n,k) = \sum_{j=1}^{m} (-1)^{(j-1)} \binom{m}{j} \binom{mn-jn}{k}$$

So, the number of combinations of k pills which include at least one pill of each type is:

$$O(m,n,k) = \binom{mn}{k} - L(m,n,k)$$

$$= \sum_{j=0}^{m} \binom{j-m-1}{j} \binom{(m-j)n}{k}$$

The number of combinations of k pills which lack pills of only one type is then $m\,O(m-1,n,k)$. Therefore the probability that a combination of *k* pills lacks exactly one type of pill is:

$$P(m,n,k) = m\,O(m-1,n,k) \Big/ \binom{mn}{k}$$

When you draw *k* pills in succession from the bottle, the probability that exactly one type will be missing amongst the first *k-1* pills and that the *k*th pill will be of the missing type is:

$$Q(m,n,k) = P(m,n,k-1) \cdot \frac{n}{mn-(k-1)}$$

So, *Q*(*m,n,k*) is a probability density over *k* (= 1 to *mn*), and the the average number of pills you must draw in succession in order to get at least one pill of each type is:

$$A(m,n) \quad = \quad \sum_{k=1}^{mn} k\,Q(m,n,k) \quad =$$

$$\sum_{k=1}^{mn}\sum_{j=1}^{m} \frac{kmn}{mn-k+1} \binom{j-m-1}{j-1} \binom{(m-j)n}{k-1} \Big/ \binom{mn}{k-1}$$

The solution to the problem (with 150 pills in the bottle: 30 each of 5 types) is $A(5,30) = \dfrac{228091951}{20821801}$ which is about 10.95.