half pills

You have a prescription to take one half of a pill per day for 20 days but the pharmacist (who is too busy to divide pills for you) gives you 10 whole pills in a bottle. On day 1, you remove a pill from the bottle, break it into two half pills, take one, and return the other half pill to the bottle. On all subsequent days you shake the bottle thoroughly and pour something out whatever comes out first - either a half pill or a whole pill; if it' s a half pill you take it and you' re done for that day; if it' s a whole pill, you split it into two half - pills, take one, and put the other back in the bottle, exactly like you did on day 1. On day 20 there can be only one half pill left in the bottle, but on day 19 there are two possibilities : either there is one whole pill or there are two half - pills left in the bottle. What is the probability that there are two half - pills in the bottle on day 19?

Solution by Michael A. Gottlieb

This is the time-continuous version of the finite difference method, which is describe d in http://www.feynmanlectures.info/solutions/half_pills_sol_3.pdf , as follows:

> If there are w whole pills and h half pills in the bottle on a given day, then the probability of drawing a whole pill equals $w/(w+h)$. One can think of this probability as the "mean whole pills" leaving the bottle by considering w and h not as integer numbers of pills but as real-valued (mean) quantities of pills. Then one can approximate that on the following day there will be, on average, $w + \Delta w$ whole pills in the bottle, with $\Delta w = - w/(w+h)$. The case for half pills is similar, but a bit more complicated because if we draw a whole pill it adds a new half pill to the bottle, while if we draw a half pill it subtracts one, so $\Delta h = w/(w+h) - h/(w+h) = (w-h)/(w+h)$.

Define $w(t)$ and $h(t)$ as real-valued functions of (continuous) time t (in days). Then we need to solve the following pair of differential equations for $w(t)$ and $h(t)$:

$$\frac{dw}{dt} = -\frac{w}{w+h}, \qquad \frac{dh}{dt} = \frac{w-h}{w+h}, \qquad (1)$$

with boundary conditions $w(0) = W$ and $h(0) = H$. The solution to the problem will then be equal to $h(t)/\big(w(t)+h(t)\big)$ for $t = 2W-2$, when $W = 10$ and $H = 0$.

First observe that when $(\mathbf{w},\mathbf{h})$ are the <u>integral</u> number of pills in the bottle on (integer) day $\mathbf{t}$, we must have $\mathbf{2w+h} = 2W+H - \mathbf{t}$, or

$$\mathbf{h} = 2W+H - \mathbf{2w} - \mathbf{t} \qquad (2)$$

because the total number of half pills in the bottle, including those "inside" the whole pills is initially $2W+H$ and one half pill is removed daily. Thus we might ask whether differential equations (1) also satisfy the (time-continuous) equation

$$h(t) = (2W + H) - 2w(t) - t. \qquad (3)$$

To see that they do first observe that

$$(w-h)/(w+h) = 2w/(w+h) - 1. \tag{4}$$

Substituting (1) into (4) we find

$$\frac{dh}{dt} = -2\frac{dw}{dt} - 1, \tag{5}$$

and integrating with respect to t yields

$$h(t) = K - 2w(t) - t. \tag{6}$$

Solving for K when $w(0) = W$ and $h(0) = H$, we find $K = 2W+H$, and thus we arrive at (3). This is a very happy thing: It shows that the solution to the differential equations (1) chosen for the time-continuous approximation will satisfy the relation (2) for the original time-discrete problem when $t = \mathbf{t}$ is an integer.

Taking the ratio of the two differential equations (1) and rearranging we have

$$\frac{dh}{dw} = h/w - 1, \tag{7}$$

from which we can solve for h as a function of w, using the boundary condition $h(W)=H$. Let $u = h/w$; multiplying both sides by w and taking derivatives with respect to w we have

$$\frac{dh}{dw} = w\frac{du}{dw} + u \tag{8}$$

Substituting the right-hand side of (8) for the left-hand side of (7) we have

$w\dfrac{du}{dw} + u = u - 1$, or

$$-du = \frac{dw}{w}. \tag{9}$$

Integration yields $-u = \log w + C$, or

$$h(w) = -w(\log w + C). \tag{10}$$

Solving for C using the initial condition $h(W)=H$, gives $C = -H/W - \log W$ and substituting this into (10) and rearranging we have

$$h(w) = w\left[H/W + \log(W/w)\right]. \tag{11}$$

To keep things simple I will now assume that $H = 0$, as per the original problem. Then, combining (11) and (3), we have

$$w\log(W\,/\,w) = 2W - 2w(t) - t. \tag{12}$$

Let $z = t - 2W$ so that

$$w\log(W/w) = -2w - z, \tag{13}$$

which can be rewritten,

$$\frac{z}{e^2 W} = (z/w)\,e^{z/w}. \tag{14}$$

Now we take the product log on both sides (a.k.a. Lambert's W function, the inverse of $f(x) = xe^x$). Noting that $z/w < 0$ and that $-1/e < z/(e^2 W) < 0$, choose the -1 branch of the product log, so $\text{plog}_{-1}((z/w)e^{z/w}) = z/w$. Then we have,

$$\text{plog}_{-1}\left(\frac{z}{e^2 W}\right) = z/w \tag{15}$$

Rearranging and substituting for z, we arrive at the solution for $w(t)$ when $H = 0$,

$$w(t) = \frac{t - 2W}{\text{plog}_{-1}\left(\dfrac{t - 2W}{e^2 W}\right)} \tag{16}$$

For $h(t)$ it is most convenient to substitute the right-hand side of (16) into (3), which gives (when $H = 0$),

$$h(t) = 2W - t + 2\frac{2W - t}{\text{plog}_{-1}\left(\dfrac{t - 2W}{e^2 W}\right)}. \tag{17}$$

Now that we have solutions for $w(t)$ and $h(t)$, we can see how good of an approximation they provide. The probability of drawing a half pill on the second to last day is $p(2W-2)$ where $p(t) = h(t)/(h(t)+w(t))$. It is interesting to note that (when $H = 0$),

$$p(t) = 1 + \frac{1}{1 + \text{plog}_{-1}\left(\dfrac{t - 2W}{e^2 W}\right)}, \tag{18}$$

so the probability of drawing a *whole* pill on day t (when $H = 0$) equals

$$-\frac{1}{1+\mathrm{plog}_{-1}\left(\dfrac{t-2W}{e^2 W}\right)} \ .$$ (19)

And, finally, here are some numerical results:

W	H	actual probability	p(2W-2)
2	0	0.500000	0.534059
3	0	0.611111	0.632203
6	0	0.715764	0.724631
10	0	0.761443	0.765907
50	0	0.838849	0.838497
75	0	0.850563	0.849858

This solution is quite accurate, even for small numbers of pills, and has the advantage (over all the other solutions I found) that the time required to calculate an answer does not depend on the number of pills.