hallway pole

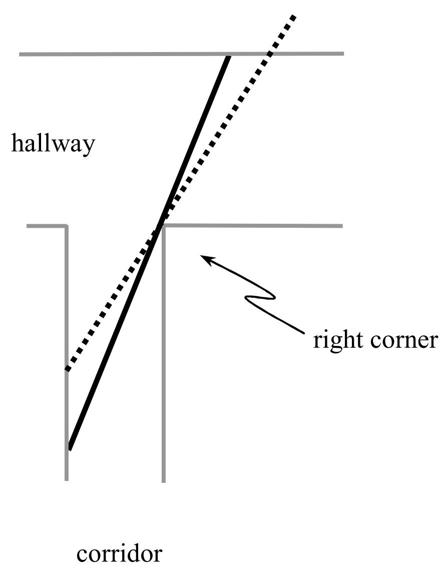
This problem comes from a calculus book by Lipman Bers:

You are an architect. Your client, living in Flatland, wants a building designed with a long 3 foot wide corridor which opens into a larger hallway. You must design the hallway for the minimum width which will allow the inhabitants to move a 24 foot-long pole down the corridor and turn it into the hallway.

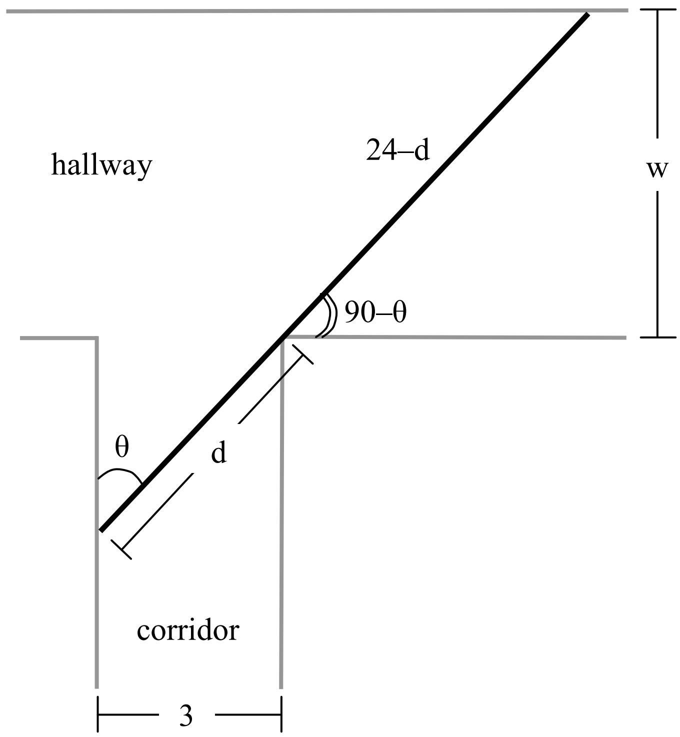
The corridor is perpendicular to the hallway. Since this is Flatland, the pole cannot be tilted up. The pole is rigid. How wide must the hallway be?

Michael A. Gottlieb's Solution

Suppose we are coming down the corridor with our 24-foot pole, and we are going to turn right at the hallway. We keep the back end of the pole against the left wall of the corridor, and the side of the pole against the right corner (see diagram below). We will get stuck if the back end of the pole can not be slid further forward along the wall of the corridor without the front end of the pole penetrating the back wall (as shown in dotted lines):



The diagram below shows a pole just barely squeezing past the corner: the pole's back end touches the left wall of the corridor, meeting it at an angle θ; the pole's front end touches the far wall of the hallway; and somewhere in the middle, the pole touches the right-hand corner–say at a distance 'd' from the pole's back end:



As can be seen from the diagram:

$$\sin(\theta) = \frac{3}{d}$$

$$\sin(90-\theta) = \frac{w}{24-d}$$

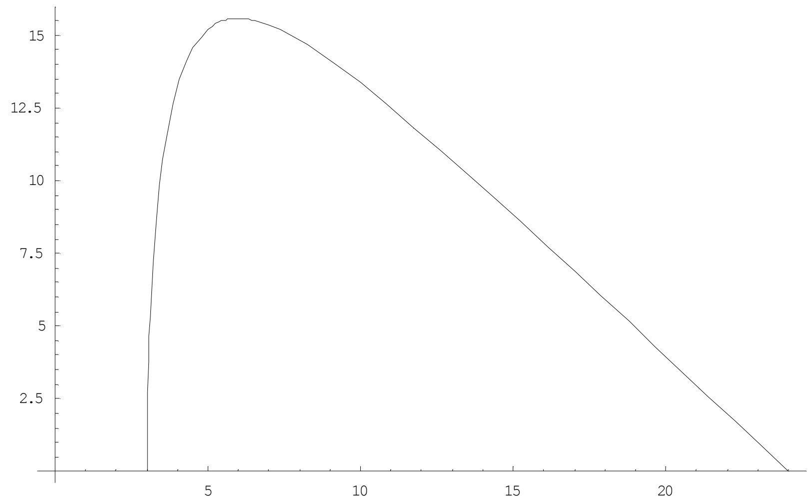
Substituting $\sin(\theta) = \cos(90-\theta)$, and using the identity $\cos(90-\theta)^2 + \sin(90-\theta)^2 = 1$, we have:

$$\left(\frac{3}{d}\right)^2 + \left(\frac{w}{24-d}\right)^2 = 1$$

Solving for *w* (positive real), we get:

$$w(d) = (24-d)\sqrt{1-\frac{9}{d^2}}$$

Here's a graph of $w(d)$ for $3 < d < 24$:



The maximum value of w is the minimum width the hallway needs to be in order to get the pole around the corner. To find this maximum, we take the derivative of $w(d)$, set it to 0, and solve for d:

$$w'(d) = \frac{216 - d^3}{d^3\sqrt{1 - \frac{9}{d^2}}} = 0$$

$$\therefore d = 6$$

To get the solution, we plug this value of d into $w(d)$:

$$w(6) = (24-6)\sqrt{1 - \frac{9}{6^2}} = 9\sqrt{3}$$

The hallway has to be at least $9\sqrt{3} \approx 15.59$ feet wide.