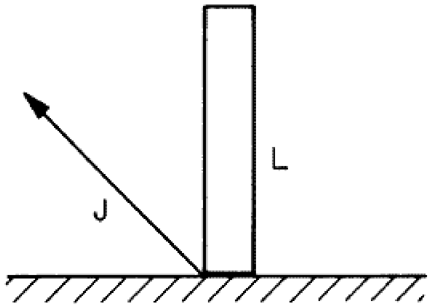
impelled rod



An upright rod of mass M and length L is given an impulse J at its base, directed at 45^o upward from the horizontal, which sends the rod flying. What value(s) should J have so that the rod lands vertically again (i.e., upright on the end at which J was applied)?

Solution by Kim Soong Ki

At time t=0 the impulse propels the rod upward with vertical velocity v_y while it rotates about it's center-of-mass with angular frequency ω. For the rod to land vertically on its base it must, during the time of flight T, rotate a total of $2\pi n$ radians (n = 1,2,3…) so that $\omega T = 2\pi n$, while it's center-of-mass, following a parabolic arc, returns to it's starting height $y(0) = y(T) = L/2$.

The vertical component of the impulse is $J \sin 45^o$, which, by the impulse - momentum theorem, must equal the rod's vertical momentum Mv_y. Thus,

$$v_y = \frac{J}{M\sqrt{2}}. \tag{1}$$

Similarly, the horizontal component of the impulse is $J \cos 45^o$, which multiplied by the length of the "lever arm" $L/2$ must equal the rod's angular momentum $I\omega$, where $I = ML^2/12$ is the moment of inertia of the rod about its center-of-mass. Thus,

$$\omega = \frac{3\sqrt{2}J}{ML}. \tag{2}$$

The height of the rod's center-of-mass at time t is $y(t) = L/2 + v_y t - ½gt^2$ (where g is the acceleration of gravity), and we must have $y(T) = L/2$, so using (1) we find that

$$T = \frac{\sqrt{2}J}{Mg}. \tag{3}$$

Since we must also have $\omega T = 2\pi n$, using (2) and (3) we conclude that

$$J = \sqrt{\frac{\pi g L n}{3}}M, \tag{4}$$

for n = 1,2,3… .