inelastic relativistic collision

A particle of mass m, moving at speed $v = 4c/5$, collides inelastically with a similar particle at rest.

(a) What is the speed v_C of the composite particle?

(b) What is its mass m_C?

Solution by Rudy Arthur:

Call the moving particle 'M', and the particle at rest 'R' (the composite particle is defined to be 'C').

The momentum of the moving particle is

$$p_M = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{4}{3}mc\,. \qquad (1)$$

And, the square of its energy is

$$E_M^2 = (mc^2)^2 + (p_M c)^2\,. \qquad (2)$$

The energy of the particle at rest is

$$E_R = (mc^2)\,. \qquad (3)$$

The square of the energy of the composite particle is

$$E_C{}^2 = (m_C c^2)^2 + (p_C c)^2\,. \qquad (4)$$

By conservation of energy: $E_M + E_R = E_C$, or squaring and rearranging,

$$2E_M E_R = E_C^2 - E_M^2 - E_R^2 \qquad (5)$$

Substituting (2) and (4) into (5):

$$2E_M E_R = \left((m_C c^2)^2 + (p_C c)^2\right) - \left(2(mc^2)^2 + (p_M c)^2\right)$$

By conservation of momentum, $p_C = p_M$, so this reduces to

$$2E_M E_R = (m_C c^2)^2 - 2(mc^2)^2$$

Squaring again:

$$4E_M^2 E_R^2 = \left((m_C c^2)^2 - 2(mc^2)^2\right)^2 \qquad (6)$$

Substituting from (2) and (3) into (6) and expanding on the right,

$$4(mc^2)^2\left((p_M c)^2 + (mc^2)^2\right) = \left((m_C c^2)^4 - 4(mc^2)^2(m_C c^2)^2 + 4(mc^2)^4\right)$$

Rearranging,

$$(m_C c^2)^4 - 4(mc^2)^2(m_C c^2)^2 - 4(mc^2)^2(p_M c)^2 = 0$$

Using (1) this reduces to

$$m_C^4 - 4m_C^2 m^2 - \frac{64}{9}m^4 = 0 \qquad (7)$$

Solving for m_C^2 (which must be positive) gives $m_C^2 = \frac{16}{3}m$, so the answer to (b) is

$$m_C = \frac{4}{\sqrt{3}}m\,. \qquad (8)$$

The momentum of the composite particle is

$$p_C = \frac{m_C v_C}{\sqrt{1-\frac{v_C^2}{c^2}}}\ . \qquad (9)$$

By conservation of momentum $p_M = p_C$, and so, substituting from (1) and (8) into (9)

$$\frac{4}{3}mc = \frac{\frac{4}{\sqrt{3}}mv_C}{\sqrt{1-\frac{v_C^2}{c^2}}} \qquad (10)$$

Solving for v_C gives the answer to (a), $v_C = \frac{c}{2}$.