maximum angle of deflection

A moving particle of mass M collides perfectly elastically with a stationary particle of mass m < M. Find the maximum possible angle through which the incident particle can be deflected.

Solution by Rudy Arthur:

Call the velocity of the moving particle $\vec{u}$ before the collision and $\vec{v}_1$ after the collision; call the velocity of the originally stationary particle $\vec{v}_2$ after the collision. We seek the maximum angle between $\vec{u}$ and $\vec{v}_1$.

By conservation of momentum: $M\vec{u} = M\vec{v}_1 + m\vec{v}_2$. (1)

By conservation of energy: $\frac{1}{2}Mu^2 = \frac{1}{2}Mv_1{}^2 + \frac{1}{2}mv_2{}^2$. (2)

Rearranging (1) gives

$$\frac{M}{m}(\vec{u} - \vec{v}_1) = \vec{v}_2 . \qquad (3)$$

Substituting into (2) gives:

$$mM(u^2 - v_1{}^2) = M^2(u^2 - 2\vec{u}\cdot\vec{v}_1 + v_1{}^2) , \qquad (4)$$

$$mM(u^2 - v_1{}^2) = M^2(u^2 - 2uv_1\cos\theta + v_1{}^2) \qquad (5)$$

where θ is the angle of deflection. Rearranging,

$$\cos\theta = \frac{u}{2v_1}(1-\frac{m}{M}) + \frac{v_1}{2u}(1+\frac{m}{M}) \qquad (6)$$

When cos θ is extremal θ is extremal, thus to find the maximum angle of deflection, differentiate the right side of (6) with respect to v_1, and solve for zero.

$$v_1 = u\sqrt{\frac{1-\frac{m}{M}}{1+\frac{m}{M}}} \qquad (7)$$

Substitute (7) into (6) and square both sides,

$$\cos^2\theta = \left(1+\frac{m}{M}\right)\left(1-\frac{m}{M}\right) \qquad (8)$$

$$\left(\frac{m}{M}\right)^2 = 1-\cos^2\theta = \sin^2\theta \qquad (9)$$

So the maximum angle of deflection is $\theta = \sin^{-1}\left(\frac{m}{M}\right)$. (10)