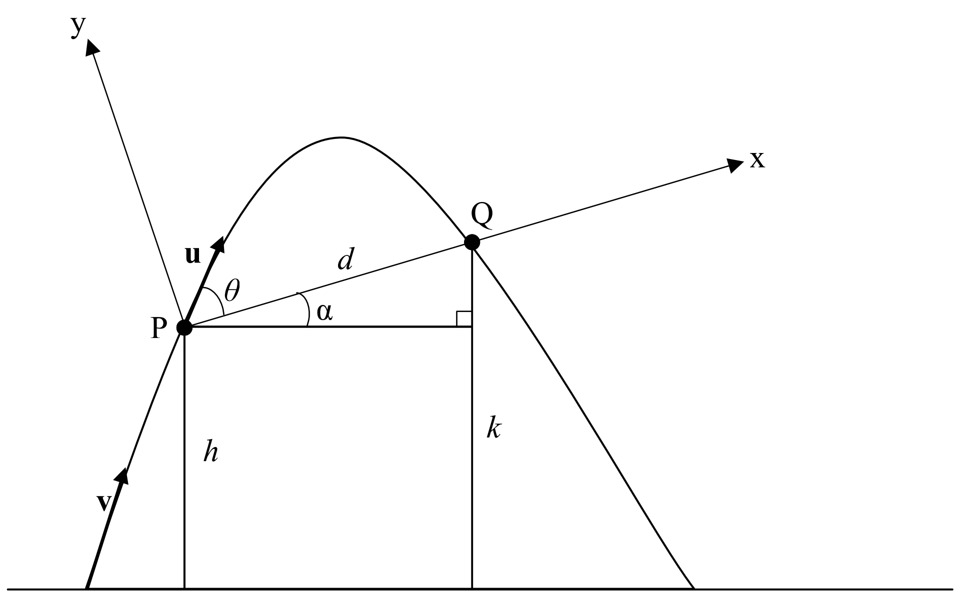
Particle points parabola

P and Q are two points a distance *d* apart at heights *h* and *k* above a given horizontal plane. What is the minimum speed v with which a particle can be projected from the horizontal plane so as to pass through P and Q?

Solution by Sukumar Chandra:



Let us assume that the particle of mass m is projected with velocity **v** from the horizontal plane and attains a velocity **u** when it is at P. Then, conservation of total mechanical energy leads to

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mgh, \text{ or } u^2 = v^2 - 2gh. \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

With P as origin, let us take an x-y axis system with PQ, which makes an angle α with the horizontal, as the x-axis. Let the velocity **u** make an angle *θ* with x-axis. Analysing the subsequent motion along the two axes, we get, at any time *t* (with the particle at P when *t*=0),

$$x = ut\cos\theta - \frac{1}{2}gt^2\sin\alpha,$$

$$y = ut\sin\theta - \frac{1}{2}gt^2\cos\alpha.$$

When the particle reaches Q, $x = d$ and $y = 0$. Hence $t = \dfrac{2u\sin\theta}{g\cos\alpha}$ and

$$d = \frac{2u^2\cos\theta\sin\theta}{g\cos\alpha} - \frac{4gu^2\sin^2\theta\sin\alpha}{2g^2\cos^2\alpha}$$

$\Rightarrow d = \dfrac{u^2}{g\cos\alpha}\sin 2\theta - \dfrac{u^2\tan\alpha}{g\cos\alpha}(1-\cos 2\theta)$, since $2\cos\theta\sin\theta = \sin 2\theta$ and $2\sin^2\theta = (1-\cos 2\theta)$.

$$\Rightarrow \frac{dg\cos\alpha}{u^2} = \sin 2\theta - \tan\alpha + \tan\alpha\cos 2\theta$$

$$\Rightarrow \sin 2\theta + \tan\alpha\cos 2\theta = \tan\alpha + \frac{dg\cos\alpha}{u^2}$$

$$\Rightarrow A\sin(2\theta+\beta) = \tan\alpha + \frac{dg\cos\alpha}{u^2}$$

taking $A\cos\beta = 1$ and $A\sin\beta = \tan\alpha$, so that $\tan\beta = 1$ and $A^2 = \sec^2\alpha$.

Hence, $\sin(2\theta+\beta) = \dfrac{1}{A}\left(\tan\alpha + \dfrac{dg\cos\alpha}{u^2}\right)$.

Since the magnitude of the sine of any angle is less than or equal to one, $\left|\dfrac{1}{A}\left(\tan\alpha + \dfrac{dg\cos\alpha}{u^2}\right)\right| \le 1$,

or, $\dfrac{1}{\sec\alpha}\left(\tan\alpha + \dfrac{dg\cos\alpha}{u^2}\right) \le 1$, because the magnitude of A is $\sec\alpha$.

Hence, $\sin\alpha + \dfrac{dg\cos^2\alpha}{u^2} \le 1$

$$\Rightarrow u^2 \ge dg\frac{\cos^2\alpha}{1-\sin\alpha}$$

$\Rightarrow v^2 - 2gh \ge dg\dfrac{1-\sin^2\alpha}{1-\sin\alpha}$, from (1)

$\Rightarrow v^2 - 2gh \ge dg(1+\sin\alpha)$, since $\sin\alpha \ne 1$

$$\Rightarrow v^2 \ge g(2h + d + d\sin\alpha)$$

$\Rightarrow v^2 \ge g(2h + d + (k-h))$, since $d\sin\alpha = k - h$

$$\Rightarrow v \ge \sqrt{g(h+d+k)}$$

Hence, $v_{\min} = \sqrt{g(h+d+k)}$.