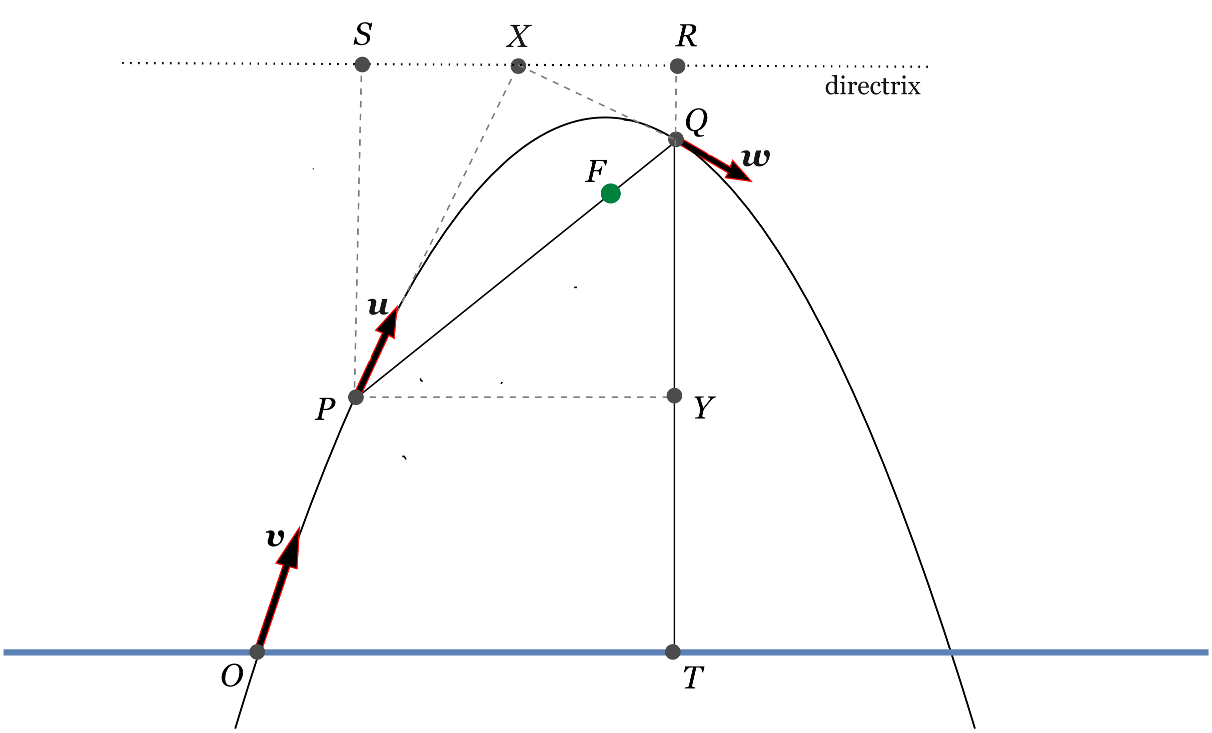
particle points parabola

P and *Q* are two points a distance *d* apart at heights *h* and *k* above a given horizontal plane. What is the minimum speed *v* with which a particle can be projected from the horizontal plane so as to pass through *P* and *Q*?

Solution by Michael A. Gottlieb



Let $\boldsymbol{u}$ be the velocity of the particle at P, let $\boldsymbol{w}$ be the velocity of the particle at Q and let the acceleration of gravity be $\boldsymbol{g}$.

For $u = |\boldsymbol{u}|$ to be the minimum speed required for the particle to get from point P to Q, it must be just fast enough so that the component of the particle's velocity in the direction of $\boldsymbol{u}$ equals zero when it arrives at Q. Thus $\boldsymbol{w}$ must be perpendicular to $\boldsymbol{u}$.

A parabola is defined as the locus of points equidistant from a fixed point (the focus) and a straight line (the directrix). It can be shown that perpendicular tangents to a parabola meet on its directrix, while the chord connecting the tangent points includes the focus. Thus the intersection X of the lines colinear with $\boldsymbol{u}$ and $\boldsymbol{w}$ lies on the directrix of the particle's parabolic path from P to Q, whose focus F lies on the chord PQ, as shown in the figure.

From the definition of parabola $PF = PS$ and $QF = QR$, while by construction $PS = RY$, $PF + QF = PQ$, and $QR + QY = RY$. It follows that

$$PS = RY = (PQ + QY)/2.$$

For the u-component of the particle's velocity to be zero when it reaches Q, kinematics dictates that $u^2 = 2g_u\, PX$, where g_u is the component of $\boldsymbol{g}$ in the direction of $\boldsymbol{u}$. Observing that $g_u = g \sin \measuredangle SXP = g\, PS/PX = g(PQ + QY)/2PX$, we find that

$$u^2 = 2\,[g(PQ + QY)/2PX]PX = g(PQ + QY).$$ [1]

By conservation of energy $v^2 = u^2 + 2g\, YT$. Thus $v^2 = g(PQ + QY + 2\, YT)$. By the problem statement $PQ = d$, $YT = h$, and $QY + YT = k$. Therefore $v^2 = g(d + h + k)$.

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[1] Another way to show it: Kinematics dictates that $w^2 = 2g_w\, QX$, where g_w is the component of $\boldsymbol{g}$ in the direction of $\boldsymbol{w}$. $g_w = g \sin \measuredangle RXQ = g\, QR/QX$, so $w^2 = 2g\, QR$. By conservation of energy $u^2 = w^2 + 2g\, QY$. Thus $u^2 = 2g\,(QR + QY) = 2g\, RY = g(PQ + QY)$.