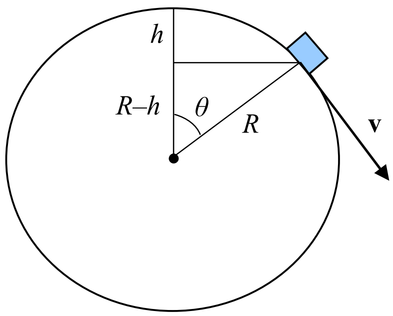
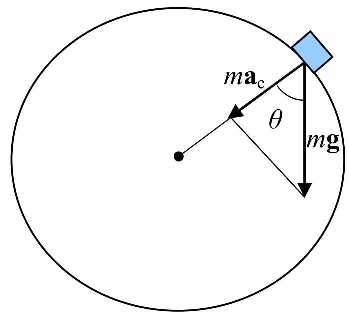
particle on sphere

A particle starts from rest at the top of a frictionless sphere of radius R and slides on the sphere under the force of gravity. How far below its starting point does it get before flying off the sphere?

Solution by Gert Hamacher

Let the distance in question be h, the mass of the particle be m, the centripetal acceleration of the particle be $\mathbf{a}_c$, and the velocity of the particle be $\mathbf{v}$. With these definitions the centripetal force is $m\mathbf{a}_c$ and the gravitational force is $m\mathbf{g}$.





During the slide down to the point where the particle flies off the sphere, there is a normal force on the particle from the sphere whose magnitude equals the radial component of the gravitational force minus the centripetal force, $mg\cos\theta - ma_c$. The normal force gets smaller and smaller as the particle speeds up, and at the point where the particle leaves the sphere, the normal force is zero, so that

$$mg\cos\theta - ma_c = 0,$$

$$\cos\theta = \frac{a_c}{g}.$$

Since with no friction the total energy is conserved,

$$\Delta E_{\text{total}} = \Delta E_{\text{kinetic}} + \Delta E_{\text{potential}} = \tfrac{1}{2}m\cdot v^2 - m\cdot g\cdot h = 0,$$

$$v^2 = 2g\cdot h,$$

and since $a_c = v^2/R$,

$$\cos\theta = \frac{a_c}{g} = \frac{v^2}{R\cdot g} = \frac{2g\cdot h}{R\cdot g} = \frac{2h}{R}.$$

By the geometry of the situation

$$\cos\theta = \frac{R-h}{R},$$

so

$$\frac{R-h}{R} = \frac{2h}{R},$$

and therefore

$$h = R/3.$$

We see that the solution is independent of the mass of the particle and of the acceleration due to gravity. So the solution is true for any particle on any planet ☺.