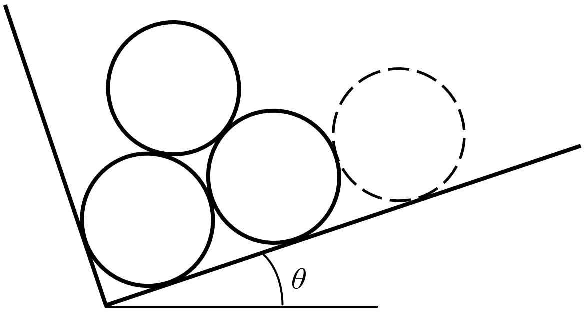
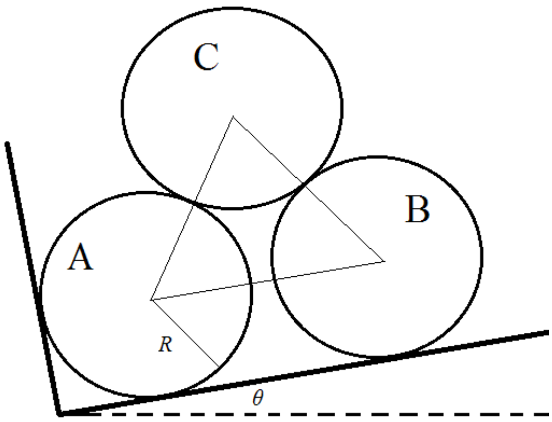
three logs

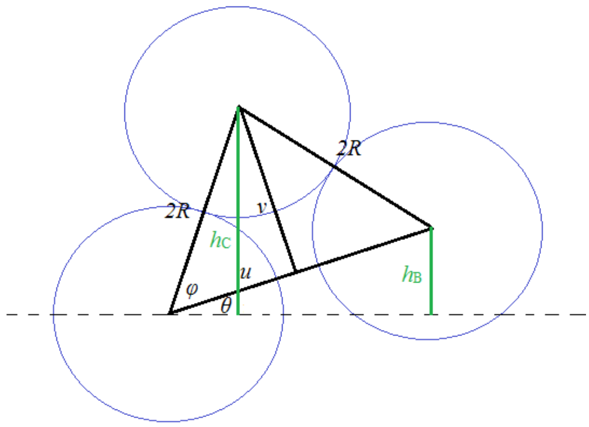


Smooth, identical logs are piled in a stake truck. The truck is forced off the highway and comes to rest on an even keel lengthwise but with the bed at an angle θ with the horizontal. As the truck is unloaded, the removal of the log shown dotted leaves the remaining three in a condition where they are just ready to slide, that is, if θ were any smaller, the logs would fall down. Find θ.

Solution by Maciej Jarocki

First, let us redraw the figure in a more convenient way. (Letter u denotes half the distance between the centers of mass of logs A and B.)





An isolated conservative system that is initially at rest (and therefore has no kinetic energy), will evolve, if possible, so that its potential energy V is converted into kinetic energy and thus decreased, or otherwise remains unchanged (i.e. in equilibrium). An increase in potential energy is not possible as it would require external work to be done on the system. Small variations in the generalized coordinate u, where $R \le u \le 2R$, cause small changes in the potential energy:

$$\delta V \approx \frac{dV}{du} \delta u$$

As explained above the system must evolve so that $\delta V \leq 0$, which gives three possibilities:

1. $\dfrac{dV}{du} = 0$, which implies $\delta u = 0$,

2. $\dfrac{dV}{du} < 0$, which implies $\delta u > 0$ if possible, otherwise $\delta u = 0$

3. $\dfrac{dV}{du} > 0$, which implies $\delta u < 0$ if possible, otherwise $\delta u = 0$.

Initially $u = R$, and thus δu can be positive. However, assuming the logs are rigid and impermeable, δu can never be negative. This means that for $dV/du > 0$ there is no possible time-evolution of the system other than to remain in equilibrium, or, to put it another way, $\delta u = 0$ is enforced at all times. Therefore, given the initial conditions (the system being at rest, and $u = R$), equilibrium will occur for $dV/du \geq 0$, where the derivative must be evaluated at $u = R$.

The height of log A cannot change, so we are only interested in how variations in u affect the total potential energy of logs B and C. Since the logs have equal mass, their total potential energy is proportional to the sum of the heights of their centers-of-mass $h_B + h_C$ (see figure), so we might as well define our potential to be $V \equiv h_B + h_C$.

For log B we have,

$$h_B = 2u\sin\theta,$$

and for log C,

$$h_C = 2R\sin(\theta+\varphi) = 2R(\cos\varphi\sin\theta + \sin\varphi\cos\theta).$$

Because logs A and B always remain in contact with log C,

$$(2R)^2 = u^2 + v^2, \quad \sin\varphi = \frac{v}{2R} \text{ and } \cos\varphi = \frac{u}{2R},$$

so the height of the center-of-mass of log C is

$$h_C = u\sin\theta + v\cos\theta = u\sin\theta + \sqrt{4R^2 - u^2}\cos\theta.$$

Therefore the potential is

$$V \equiv h_B + h_C = 3u\sin\theta + \sqrt{4R^2 - u^2}\cos\theta.$$

Taking the derivative we obtain

$$\frac{dV}{du} = 3\sin\theta - \frac{u}{\sqrt{4R^2 - u^2}}\cos\theta,$$

which evaluated at $u = R$ yields the condition

$$3\sin\theta - \frac{\cos\theta}{\sqrt{3}} \geq 0.$$

Therefore the smallest angle is $\theta_{\min} = \tan^{-1}\left(\dfrac{1}{3\sqrt{3}}\right)$.